Charlotte Elisabeth Ameil

Charlotte Ameil

(livre 1)

Infinite Dimensional Folder-Space

elite Project Book

Charlotte Ĕlisabeth Ameil - Mathematiques

[Les Vecteurs]

Welcome to Vector Space!

Notation. Defining $B\Delta$ to be the grid of points located on the lower boundary B, and $R\Delta$ to be the collection of grid points in the rectangle R. Here B and R are sketched along with the associated lower boundary. we define the 'discrete rectangle'

$$R\Delta = \{(x_j, t_m) : x_j \in [0,1],\ t_m \in [0,T]\}$$

and the lower boundary

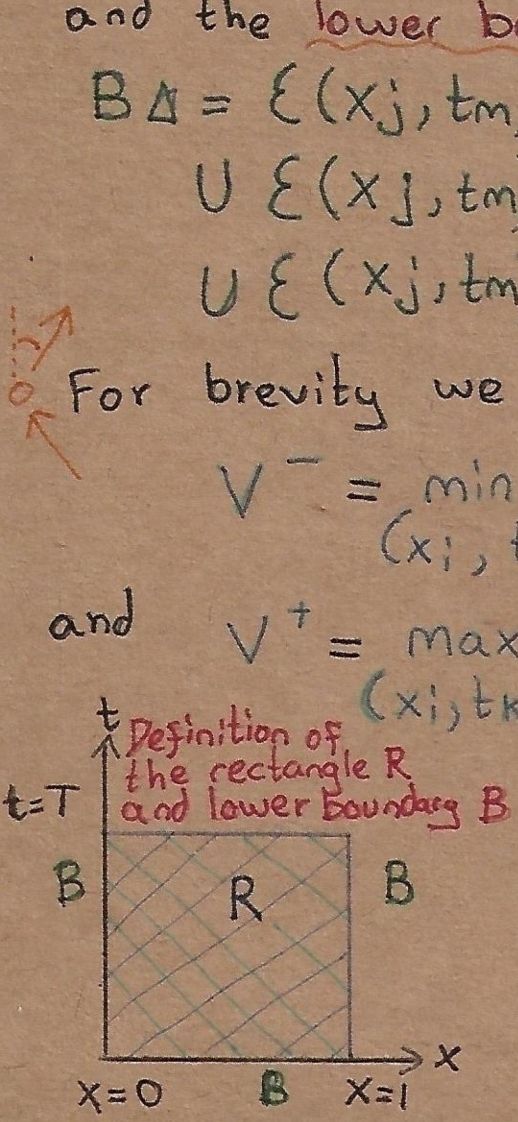
$$B\Delta = \{(x_j, t_m) : x_j = 0, 0 \leq t_m \leq T\}$$
$$\cup \{(x_j, t_m) : t_m = 0, 0 \leq x_j \leq 1\}$$
$$\cup \{(x_j, t_m) : x_j = 1, 0 \leq t_m \leq T\}$$

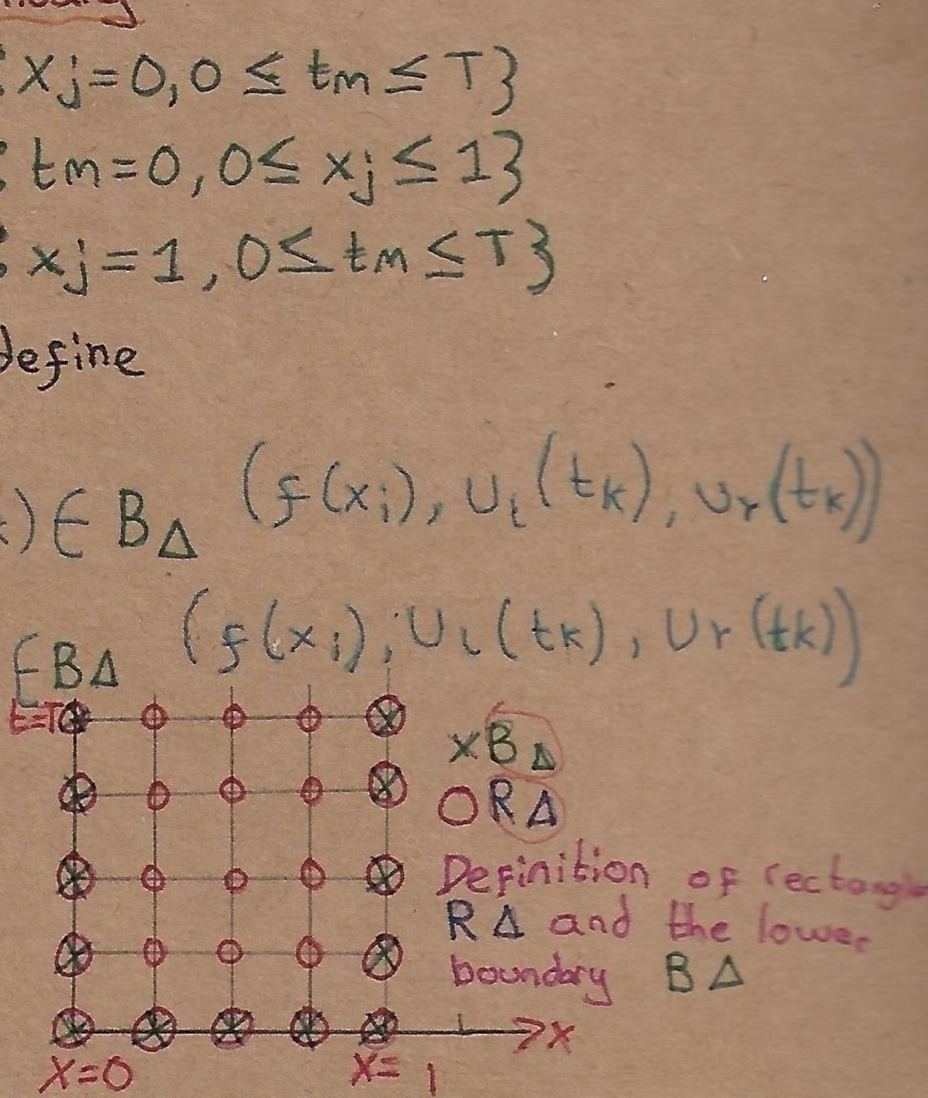
For brevity we define

$$V^- = \min_{(x_i, t_k) \in B_\Delta} (f(x_i), u_l(t_k), u_r(t_k))$$

and

$$V^+ = \max_{(x_i, t_k) \in B\Delta} (f(x_i), U_l(t_k), U_r(t_k))$$





Vector Space Definitions - Part 1 Vectors

Definition Vector Space

A vector space V (aslo called a linear space) is a non-empty set of elements $x, y, z \dots$ called vectors for which there are defined two algebraic operations.

One of these, called vector addition, is a rule for associating with each pair of vectors x and y in V another vector $x+y$ called the sum of x and y.

The other, called multiplication of a vector by a scalar, is a rule for associating with each vector x and each scalar λ a vector λx called the scalar multiple of x.

The following axioms are assumed to be true, with x, y, z representing any vectors in V. And λ and μ any scalears, either real or complex.

Axioms

Addition

Addition

$+$

A.1 $x+y$ is a vector in V (closure under addition)

A.2 $x+y = y+x$ (commutativity)

A.3 $x+(y+z) = (x+y)+z$ (associativity)

A.4 There is a unique zero or null vector 0 in V with the property that
$x + 0 = x$

A.5 Each vector x in V has associated with it a unique element $-x$ called the negative (additive inverse) of x with the property that $x + (-x) = 0$.

Multiplication

In all of these - if λ and μ are real numbers then we have a real vector space otherwise it is called a complex vector space

multiplication

$\times$

M.1 λx is in V (closure under scalar multiplication)

M.2 $\lambda(x+y) = \lambda x + \lambda y$ (distributivity)

M.3 $(\lambda + \mu)x = \lambda x + \mu x$ (distributivity)

M.4 $\lambda(\mu x) = (\lambda\mu)x$ (associativity)

M.5 If $\lambda = 1$
then $1x = x$ (scaling by unity)

Vector Space Definitions - Continued - Part 2 | Vectors

A comparison of the previous ten axioms with properties P.1 to P.8 possessed by geometrical vectors in $\mathbb{R}^3$, shows the inclusion of the closure axioms A1 and M1.
These ensure that in a general vector space the result of addition and multiplication is to produce vectors which themselves belong to the same vector space. And do not lie outside it.
A formal statement of these properties was unnecessary in the case of the vector space of geometrical vectors in $\mathbb{R}^3$ - since they were included in the definitions of addition and scalar multiplication of geometrical vectors.

Let $\lambda_1, \lambda_2, \ldots \lambda_m$ be any m scalars and $x_1, x_2, \ldots, x_m$ any m vectors beloning to V.

Then the sum

$$\left[\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m\right] \qquad (1)$$

Which by A.1 also belongs to V, is called a linear combination of the vectors $x_1, x_2, \ldots, x_m$.
If the set of vectors $x_1, x_2, \ldots x_m$ is fixed, but the scalars $\lambda_1, \lambda_2, \ldots \lambda_m$ are allowed to take all possible values, a set U of vectors is generated containing all the resulting linear combinations (1).
The set U, itself a vector space, is said to be spanned by the vectors $x_1, x_2, \ldots, x_m$, which are themselves called the span of U and written span $\{x_1, x_2, \ldots, x_m\}$.

Linearly Independent

As a set of vectors $x_1, x_2, \ldots, x_m$ will be said to be linearly independent if the vector equation

$$\left[\lambda_1 x_1 + \lambda_2 x_2 + \ldots \lambda_m x_m = 0\right]$$

Linearly Dependent

is only true when the scalars $\lambda_1 = \lambda_2 = \ldots \lambda_m = 0$. If the vectors $x_1, x_2, \ldots x_m$ are such that (2) is true for some set of scalars $\lambda_1, \lambda_2, \lambda_m$ - not all zero then the vectors $x_1, x_2 \ldots x_m$ are Linearly Dependent

A set of vectors $e_1, e_2, \ldots, e_n$ belonging to V will be called a basis for the vector space V when it spans V and is linearly independent.
The dimension n of a vector space V is the number of vectors in a basis for V.
For a vector space V of dimension n, this definition implies that every set of m vectors in V will be linearly dependent if $m > n$.

- If the dimension of a vector space is finite the space is called a finite dimensional vector space.
- If however, the vectors needed to form a basis are infinite in number the associated vector space is called an infinite dimensional vector space.

- The importance of the dimension of a space lies in the fact that it may be proved that all finite dimensional vector spaces with the same dimension have identical algebraic properties.

As a result the study of the simplest space with dimension n provides information about the structure of all other spaces of equal dimension.
The preservation of algebraic properties in this way among all vector spaces of the same dimension is called an isomorphism.
This means, for example, that all vector spaces of dimension 3 are isomorphic to the vector space of all geometrical vectors in $\mathbb{R}^3$.

A set of vectors U will be called a subset of the vector space V if every element of U belongs to V. It will be called a proper subset of vector space V if all elements of U belongs to V, but V also contains at least one element which does not belong to U.

- Thus V is a subset of itself but not a proper subset of itself. (all elements in V belong to V)

Examples of finite dimensional vector spaces

① ○ ① The space V containing all geometrical vectors in $\mathbb{R}^3$ is a real vector space with vector addition and the multiplication of a vector by a scalar defined in the first page.

The space is of dimension 3 because i, j and k form a basis for all such vectors.

The vectors $a = i$, $b = i + 2j$ and $c = i + j + 3k$ are linearly independent, and so also form a basis for $\mathbb{R}^3$, because the only solution to

$$\left[\lambda_1 a + \lambda_2 b + \lambda_3 c = 0\right]$$

is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. However, the vectors $a = i + 5j - 5k$, $b = i + j + k$ and $c = 2j - 3k$ are linearly dependent because $[a = b + 2c]$.

Dimension

Vectors b and c thus span a subspace of $\mathbb{R}^3$ of dimension 2 (a plane of $\mathbb{R}^3$)

② ○ ② Let V be the set of vectors X comprising all ordered number pairs $x = (a, b)$ with a, b real numbers. Let $x = (a, b)$, $y = (c, d)$ be any two vectors in V, and define vector addition by the rule

$$\left[x + y = (a + c, b + d)\right]$$

and the product λx, with λ any real number, by the rule

$$\left[\lambda x = (\lambda a, \lambda b)\right]$$

It is easily checked that the vectors in V satisfy axioms A and M, which shows that V is a vector space.

Dimension

We may take as a basis for this vector space the vectors

$$e_1 = (1, 0) \text{ and } e_2 = (0, 1)$$

because every vector $x = (a, b)$ may be written $x = ae_1 + be_2$ showing the dimesion of this space is 2.

③ As a final example, let the space V be the set of vectors X comprising all ordered n-tuples of real numbers $x = (x_1, x_2, \ldots, x_n)$. Let $x = (x_1, x_2, \ldots, x_n)$ and let $y = (y_1, y_2, \ldots, y_n)$ be any two vectors in V, and define vector addition by the rule

$$\left[x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\right]$$

and the product λx with λ any real number by the rule

$$\left[\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)\right]$$

It is a straightforward matter to verify that the vectors in V satisfy axioms A and M, which shows that V is a vector space. We may take as a basis for this n-dimensional space the vectors $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, ..., $e_n = (0, 0, 0, \ldots, 1)$, since every vector x in V may be expressed as a linear combination of $e_1, e_2, \ldots, e_n$.

The space V involved in this example is a direct generalization of the vector space of geometrical vectors in $\mathbb{R}^3$ to an analogous space of vectors in $\mathbb{R}^n$.

Dot Product Revisited

A specially useful class of vector spaces is that in which it is possible to associate with each pair of vectors x, y in the space V a real number, written (x, y), called the inner product of x and x. Or the dot product (sometimes the scalar product). When the below are met we have a real inner product space.

Linearity

P_1 - For all x, y, z in V and all scalars λ, μ
$(\lambda x + \mu y, z) = \lambda(x, z) + \mu(y, z)$ (Linearity of inner product) P_1

Symmetry

P_2 - For all x, y in V
$(x, y) = (y, x)$ (Symmetry of inner product) P_2

Positive Definiteness

P_3 - For every x in V
$(x, x) \geq 0$, with $(x, x) = 0$ if, and if, $x = 0$ (Positive definiteness of inner product) P_3

Finite Dimensional Inner Product Space

- The vector space $\underset{\sim}{V}$ with vectors x comprising of all ordered n-tuples of real numbers $x = (x_1, x_2, x_3, \ldots, x_n)$, with the operations of addition and scalar multiplication defined in the previous pages example - becomes an inner product space when the inner product is defined as:

$$\left[(x,y) = x_1 y_1 + x_2 y_2 + x_3 y_3 \ldots x_n y_n\right]$$

→ To establish this assertion it is necessary to verify that the inner product just defined possesses properties of P_1 to P_3.

Linearity of Inner Product

P_1 Linearity Of The Inner Product

- By definition

$$\left[\lambda x + \mu y = (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2 \ldots \lambda x_n + \mu y_n)\right]$$

and so from the definition of the inner product

$$(\lambda x + \mu y, z) = (\lambda x_1 z_1 + \mu y_1 z_1, \lambda x_2 z_2 + \mu y_2 z_2, \lambda x_3 z_3 + \mu y_3 z_3 \ldots, \lambda x_n z_n + \mu y_n z_n)$$

$$= \lambda(x_1 z_1, x_2 z_2, x_3 z_3 \ldots n)$$

$$\mu(y_1 z_1, y_2 z_2, y_3 z_3 \ldots y_n z_n)$$

$$= \left[\lambda(x,z) + \mu(y,z)\right]$$

Symmetry of the Inner Product

P_2 Symmetry of the Inner Product

- From the definition of the inner product

$$(x,y) = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots x_n y_n$$

$$= y_n x_n + \ldots, y_3 x_3 + y_2 x_2 + y_1 x_1 = (y, x)$$

Positive Definiteness

P_3 Positive Definiteness

- From Inner Product definition $(x,x) = x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2$ Showing (x,x) is a sum of squares. Thus $(x,x) > 0$ if $x \neq 0$, and $(x,x) = 0$ if $x = 0$. Showing (x,x) is a positive definite.

Finite Dimensional Inner Product Space — Vectors

Norm / Length

The inner product may be used to introduce the notion of a length into a general inner product space V be defining the norm $\|x\|$ of a vector x in V as:

$$\boxed{\|x\| = \sqrt{(x,x)}}$$

which from P.3 is seen to be a non-negative number. Only the zero vector $x=0$ has a zero norm.

Checking that a norm is distance

To see that the norm exhibits the properties of a distance we first recall that if $d(x,y)$ is the euclidean distance between points with geometrical position vectors x and y in $\mathbb{R}^3$, then d satisfies the axioms-

- D.1 $d(x,y) > 0$ for $x \neq y$
  $d(x,y) = 0$ if $x = y$ (distance is positive)
- D.2 $d(x,y) = d(y,x)$ (distance is symmetric)
- D.3 $d(x,z) \leq d(x,y) + d(y,z)$ (distance = triangle inequality)

The norm already exhibits properties D.1 an D.2 by virtue of its definition. Property D.3 follows by using the definition of the norm and the axioms of an inner product space to prove that.

$$\boxed{\|x+y\| \leq \|x\| + \|y\|}$$

- The norm $\|x\| = (x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2)^{1/2}$ associated with the inner product space of Example 2 is called the euclidean norm, because when $n=3$ it reduces to an ordinary distance in $\mathbb{R}^3$.
- Another inequality involving norms which follows directly from the definition of the norm and the axioms of an inner product space is:

Chauchy Schwarz

$$\boxed{|(x,y)| \leq \|x\| \, \|y\|}$$

- which is the Cauchy-Schwarz Inequality.

Infinite Dimensional Inner Product Space

Vectors

Continous Functions

Finally we mention the inner product space of continous functions ~ $f(x)$, $g(x)$ ...., defined on an interval $a \leq x \leq b$ with ordinary definitions of addition and multiplication by a scalar, in which the inner product is defined by the definite integral -

$$\boxed{(f,g) = \int_a^b w(x) f(x) g(x)\, dx}$$

with $w(x) \geq 0$ a given weight function.

Properties P_1 to P_3 required an inner product follow directly from this definition and from the properties of the definite intregral.

This form of inner product is used in many areas of math, such as the weight function $w(x) = 1$, and $f(x) = \sin(m\pi x/L)$, $g(x) = \sin(n\pi x/L)$ for $m, n = 1, 2, 3, \ldots$

The elements or vectors, in this inner product space are the functions $\sin(\pi x/L)$, $\sin(2\pi x/L)$, $\sin(3\pi x/L)$ ..... These vectors are mutually orthogonal over the interval $0 \leq x \leq L$, in the sense that the inner product of two different vectors is zero, because of the elementary result

$$\left(\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right) = \begin{cases} \int_0^L \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = 0 & \text{for } m \neq n \\ \frac{L}{2} & \text{for } m = n \end{cases}$$

General case for Continous functions

If we take $w(x) = 1$, and general case where the functions are continous on the interval $a \leq x \leq b$, then by setting $x = f(x) - h(x)$ and $y = h(x) - g(x)$ the triangle inequality takes:

$$\left(\int_a^b [f(x) - g(x)]^2 dx\right)^{1/2} \leq \left(\int_a^b [f(x) - h(x)]^2 dx\right)^{1/2} + \left(\int_a^b [h(x) - g(x)]^2 dx\right)^{1/2}$$

and the cauchy-schwarz inequality becomes:

$$\left|\int_a^b f(x) g(x)\, dx\right| \leq \left(\int_a^b [f(x)]^2 dx\right)^{1/2} \left(\int_a^b [g(x)]^2 dx\right)^{1/2}$$

Scalar Products and Equations of planes | Vectors

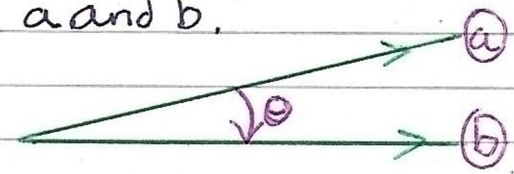
We can add vectors together, and we can multiply a vector by a scalar. But so far we have not 'multiplied' a vector.
In fact there are two kinds of vector 'product'. In both cases the element of multiplication is only a part of the story.

Definition 1

The scalar product of two vectors a and b is defined to be

$$a.b = |a|\,|b| \cos\theta$$

where θ is the angle between the directions of the vectors a and b.



The name of the scalar product refers to the fact that the result of a scalar product is a 'scalar'. It is sometimes called the dot product because of the dot expression. Which should always be clearly shown to tell us what sort of product this is.

Length of a Vector

Since $\cos(0) = 1$

$$a.a = |a|\,|a| = |a|^2$$

Which we shall find useful in many types of examples with vectors.

Note that, if θ is acute then a.b is positive. If θ is obtuse then a.b is negative. If θ is a right angle then a.b is zero.

Orthogonal Vectors

If vectors a and b are orthogonal, then from the previous examples we know that $a.b = 0$.
However, $a.b=0$ does not nesserily imply that the vectors are perpendicular.
Since one of the vectors could be the zero vector, in which case the angle cannot be defined.

So we should say that if $a.b=0$, then $a=0$, or $b=0$, - or a and b are orthogonal.

$$\text{Since } \cos(-\theta) = \cos(\theta)$$
$$b.a = |b|\,|a| \cos(-\theta) = |a|\,|b| \cos(\theta) = a.b$$

and this shows that the scalar product operation is 'commutative'.

The scalar behaves like a product in the sense it is distributive over addition. That is:

$$a.(b+c) = a.b + a.c$$
$$(b+c).a = b.a + c.a$$

This is particulary useful when we express vectors in terms of the standard unit vectors i, j and k.

Consider $\quad a = a_1 i + a_2 j + a_3 k$
and $\quad b = b_1 i + b_2 j + b_3 k$

So $$a.b = a_1 j . b_1 i + a_2 j . b_1 i + a_3 k . b_1 i$$
$$+ a_1 i . b_2 j + a_2 j . b_2 j + a_3 k . b_2 j$$
$$+ a_1 k . b_3 k + a_2 j . b_3 k + a_3 k . b_3 k$$

Now $\alpha i . \beta j = 0$ whatever the values α and β since the vectors are orthogonal. Also $\alpha i . \beta i =$
$$|\alpha i| \, |\beta i| \cos\theta = \alpha\beta \text{ etc..}$$

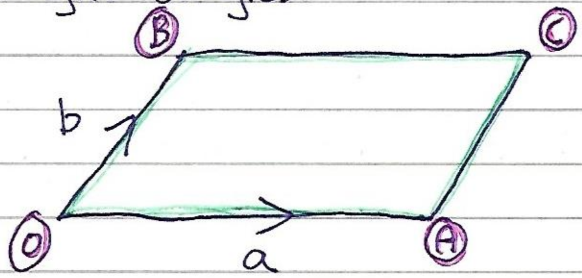
So that
$$a.b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note that
$$a.a = a_1^2 + a_2^2 + a_3^2$$

which confirms the result given previously.

Ex(1) Example

Given that OACB is a parallelogram, we will show it is a rhombus if and only if its diagonals are at right angles.



Solution

Since we are told that OACB is a parallelogram, we know that the opposite sides are both parallel, and equal in length.

So that if $OA = a$, and $OB = b$
then $BC = a$, and $AC = b$.

The diagonals of the parallelogram are OC, BA.

From the triangles in the diagram we can see that

$$BA = a - b \quad \text{and} \quad OC = a + b$$

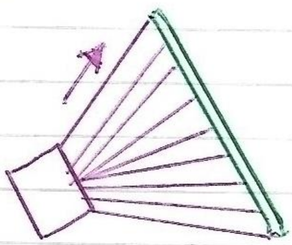
Then — $BA . OC = (a-b) . (a+b)$
$= a.a + a.b - b.a - b.b$
$= |a|^2 - |b|^2$

Scalar Products and Equations of Planes Vectors

(continued) So $BA.OC = 0$ if $|a| = |b|$, which means that the diagonal are perpendicular if all the sides are equal in length.
That is, only if the parallelgram is a rhombus.

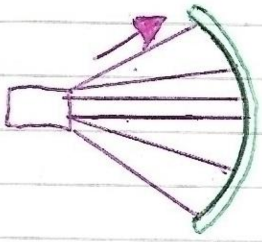
Projections and Components

When you see a film, or slides being shown, the pictures are projected onto a flat screen.
In general the clearest images are obtained when the rays meeting the screen are as close to being perpendicular to the surface as possible.



When showing slides it is usual to have the line joining the centre of the picture on the screen to the lens of the projector, at right angles to the screen.

In wide-angle films, to prevent the distortion a flat-screen would produce, A curved screen is used.

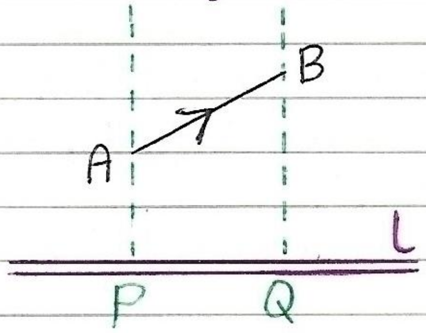


The projector is placed on the axis of the cylinder in which the screen is.
So that all the horizontal rays from the projector meet the screen orthogonally. In this way a better defined picture is acheived.

In this section only orthogonal projections onto a plane shall be considerd.

Orthogonal Projections onto a Plane — Vectors

- Consider first the two-dimensional case below:



This figure illustrates the orthogonal projection PQ of the line segment AB onto the line L.

The orthogonal projection is set to all points X on L, where the perpendicular to L at X meets the line segment AB.

Another way of looking at this is by considering the projection of AB onto L to be the shadow of AB on L. Produced by shining a parallel beam of light orthogonal to L from a position with AB between the source of light and the line L.

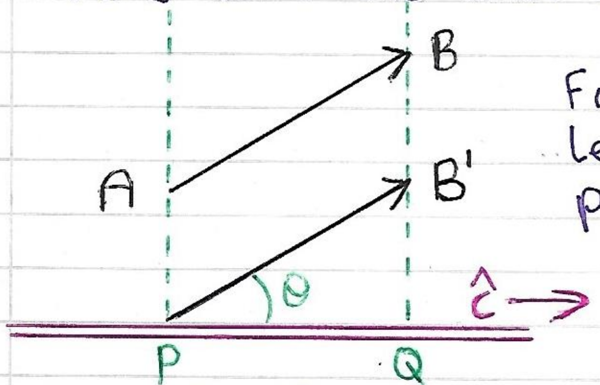
An Alternative: If instead of a straight line AB we have a curve between A and B, we should get the same projection. This is provided the curve did not exceed beyond the dashed lines PA and QB perpendicular to L.

- The three-dimensional case is simular. Now instead of a line 'L', we can have a plane π. The projection of the curve AB onto the plane is set to all points for X in π for which the line orthogonal to π at X meets the curve AB.
  Again if we think of the plane as being horizontal, the projection of AB onto the plane is the shadow of AB on the plane in a vertical beam of light.

It is possible to project a curve onto a line in $\mathbb{R}^3$.
In this case the curve would be lying in entirely of a plane containing 'L', and be at right angles to π.
This idea is used many times in vectors.

We will now put this information of projections, to scalar products.

Lets take the case of two dimensions first.



From the diagram on the left, we see that AB will produce the same projection on 'L' as PB'.
(Or infact any other line segment that has the same length and is parallel to AB)

Now from triangle PB'Q comes

$$PQ = PB' \cos\theta = AB \cos\theta.$$

But if $\hat{c}$ is a unit vector in the direction of the line 'L', then

$$\hat{c} . AB = 1 . AB . \cos\theta$$

which is the projection of AB onto L.

Normally we are only interested in the length of the projection. So if the scalar product is negative we drop the minus sign.

Because in practical applications we need to know how much effort a particular vector has in a particular direction.
- The idea of projections is important.

Orthogonal Projections onto a Plane	Vectors

Definition 2

Given the vectors 'a' and 'b', we define the component of 'a' in the definition of the direction of 'b' to be $a.\hat{b}$

Example

- Findibg the component of 'a' in the direction of 'b' when

$a = 3i - 2j + k$

$b = 2i + 2j + k$

Solution

$\hat{b} = \frac{2}{3}i + \frac{2}{3}j + \frac{1}{3}k$ So that:

$a.\hat{b} = 1$ So the components of 'a' are in the direction of 'b' is 1.

Angles From Scalar Products

Because we have two definitions of 'a.b' for those vectors whose components we know, we can use the two expressions to find the angle between the vectors.

- Suppose $a = a_1 i + a_2 j + a_3 k$
  $b = b_1 i + b_2 j + b_3 k$

then $a.b = |a|\,|b| \cos\theta$

Where θ is the angle between the directions of 'a' and 'b' - and:

$$a.b = a_1b_1 + a_2b_2 + a_3b_3$$

Equating $(a.b = |a|\,|b|\cos\theta)$
and $(a.b = a_1b_1 + a_2b_2 + a_3b_3)$ we have:

$|a|\,|b|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

SO $\dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$

Angles from scalar products - Example

- Suppose that $a = 2i + 2j - k$ and $b = 2i - 3j + k$

then the angle between 'a' and 'b' is θ, where:

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

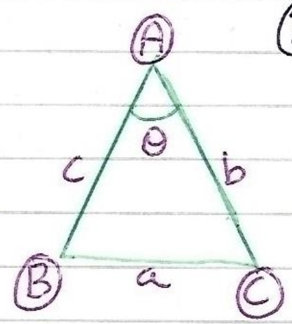
$$= \frac{2.2 + 2.(-3) + (-1).1}{\sqrt{9}\sqrt{14}}$$ so:

$$\cos\theta = \frac{-1}{\sqrt{14}} \quad (\theta \simeq 105.5^\circ)$$

The cosine Rule

The triangle of vectors rule states that the sides of triangle ABC, obey the rule

$$BC = BA + AC$$



Taking the scalar product of each side of this equation with BC we get:

$$\begin{aligned} BC.BC &= (BA + AC).BC \\ &= (BA + AC).(BA.AC) \\ &= (BA.BA + 2BA.AC + AC.AC) \end{aligned}$$

- Remember that $AB = -BA$ so that we can write the above equation as:

$$BC^2 = BA^2 + AC^2 - 2AB.AC\cos\theta$$

Where in the example on the last page -

θ is the angle between the directions of the vectors AB and AC.
With the usual convention that in triangle 'ABC', the sides opposite angle A, B, C are called respectively a, b, c.
This reduces to

$$a^2 = c^2 + b^2 - 2cb \cos A$$

Which is the standard formula for the cosine rule.

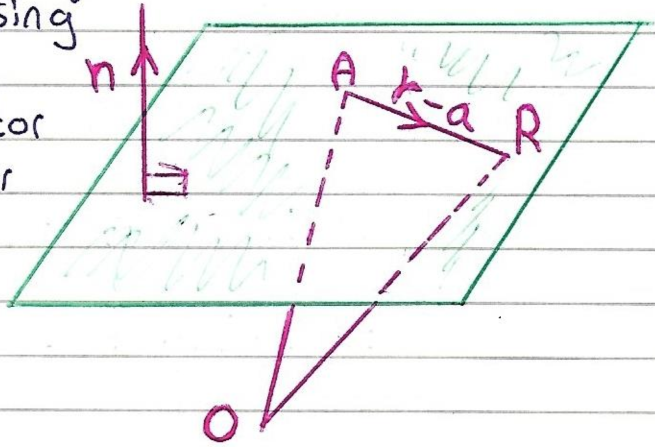
Vector Equation of a Plane

Consider a plane π, and a vector 'n' which is orthogonal to π.
Suppose A, is a fixed point on the plane, and let 'a' be the position vector of A.

Let R, with position vector 'r', be any point on the plane.
As $AR = r - a$, and since AR is a vector parallel to the plane π, and n is orthogonal to π. We have:

$$AR . n = (r - a) . n = 0$$

So the vector equation of the plane passing through the point with position vector 'a' - perpendicular to the vector 'n' is



$$(r - a) . n = 0$$

(continued) This leads to finding the cartesian equations of a plane.

Previous Diagram

- For if we think of a point 'R' in $\mathbb{R}^3$ as having coordinates (x,y,z) and position vector 'r'. A given point 'A' with coordinates (a,b,c) and position vector 'a'. With 'n' as $n = pi + qj + rk$ $(n = pi + qj + rk)$ normal to the plane. (see diagram, bottom of last page)

- We can write the cartesian equation of the plane, as follows:

$$ax + by + cz - (ap + bq + cr) = 0$$

or writing $d = ap + bq + cr$
we can express the equation of π as

$$ax + by + cz = d$$

Example

Suppose a vector normal to the plane π is $i + 2j + 2k$, and then suppose that π contains the point 'A'. Whose coordinates are $(2,1,5)$.

Finding the vector equation, and then the cartesian.

First the vector = $\left(r - \begin{pmatrix}2\\1\\5\end{pmatrix}\right) \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = 0$

and the cartesian equation is:

$$x + 2y + 2z - (2\times1 + 1\times2 + 5\times2) = 0$$

$$= x + 2y + 2z = 14$$

The Intersection of two Planes

As seen previously - the cartesian equation of a plane is

$$ax + by + cz = d$$

where at least one of a, b, c is non-zero.

Take a look now at the two planes π and π' whose equations are

$$ax + by + cz = d \quad (\text{for } \pi)$$
$$a'x + b'y + c'z = d' \quad (\text{for } \pi')$$

There's no loss of generality in supposing $a \neq 0$, since at least one of a, b, c is non-zero.

Then substituting for x from $(ax + by + cz = d)$ into $(a'x + b'y + c'z = d')$ we get, after simplification

$$\left(b' - \frac{ba'}{a}\right)y + \left(c' - \frac{ca'}{a}\right)z = d' - \frac{da'}{a}$$

Suppose the coefficients of y and z are both zero. Then the above equation only has a solution if $d' - da'/a = 0$, since otherwise we would of had an equation with zero on one side and a non-zero on the other.

This means there can be no intersection of the planes π and π' - if $b' = ba'/a$ and $c' = ca'/a$ unless $d' = da'/a$.

If all three of these equations are true then $a'x + b'y + c'z = d'$ is produced by multiplying both sides of $ax + by + cz = d$ by a'/a.

(continued) So from the previous page, this means the equations

$$ax + by + cz = d$$
$$a'x + b'y + c'z = d'$$

represent the same plane.

So in the case of π and π', they are coincident.
As $b' = ba'/a$ and $c' = ca'/a$
so $a' : b' : c' = 1 : b/a : c/a = a : b : c$

Meaning the planes π and π' are parallel, but not coincident - if and only if that

$$a' : b' : c' = a : b : c$$
and $$a' : b' : c' : d' \neq a : b : c : d$$

Looking again at the equation:

$$\left(b' - \frac{ba'}{a}\right)y + \left(c' - \frac{ca'}{a}\right)z = d' - \frac{da'}{a}$$

Suppose (not loosing generality) the coefficient of y is non-zero.
Then we have

$$y = Az + B \quad \text{and} \quad x = (z + D)$$

Where A, B. C, D are straight lines, unless:

① They are parallel - in which case
$a' : b' : c' = a : b : c$
but $a' : b' : c' : d' \neq a : b : c : d$

② They are coincident - in which case
$a' : b' : c' : d' = a : b : c : d$

Example

Finding the equation of the line of intersection of the planes

$$x - y - 2z = 3$$
and
$$2x + 3y + 4z = -2$$

Solution

From the first equation of the line of intersection of the planes $x - y - 2z = 3$

From writing $x - y - 2z = 3$ and substituting into the second equation, gives-

$$2(y + 2z + 3) + 3y + 4z = -2 \quad (\text{or}) \quad 5y + 8z = -8$$

This means $y = -\frac{8}{5}(z+1)$
So by substituting this for x gives:

$$x = -\frac{8}{5}(z+1) + 2z + 3 \quad (\text{or}) \quad x = \frac{2z+7}{5}$$

We can collect together this information giving:

$$\frac{x - \frac{7}{5}}{2} = \frac{y + \frac{8}{5}}{-8} = \frac{z}{5} \leftarrow \left(\frac{z}{5}\right)$$

Which is the set of equations of the line passing through the point $\left(\frac{7}{5}, \frac{-8}{5}, 0\right)$ having direction:

$$(2, -8, 5)$$

In vector notation, this would be written:

$$r = \begin{pmatrix} 7/5 \\ -8/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix}$$

There are some useful checks that can be used with the results of the previous pages.

First we can test if $\left(\frac{7}{5}, \frac{-8}{5}, 0\right)$ really does lie on both planes.
We can do this by substituting into both equations:

$$\frac{7}{5} - \frac{(-8)}{5} - 2 \times 0 = 3$$

and

$$2 \times \frac{7}{5} + 3 \times \frac{(-8)}{5} + 4 \times 0 = -2$$

So first check is positive.

Next we can see if the vectors 'n' and 'm' are perpendicular to the two planes.
Then since 'L' lies on both planes, its direction is perpendicular to both 'n' and 'm'.

In this case:

$$n = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

The vector $\begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix}$ is parallel to the line 'L'

and $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix} = 0$ (and) $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix} = 0$

This verifies that we have the correct direction for the line of intersection, and this second check is complete.

The Intersection of three Planes

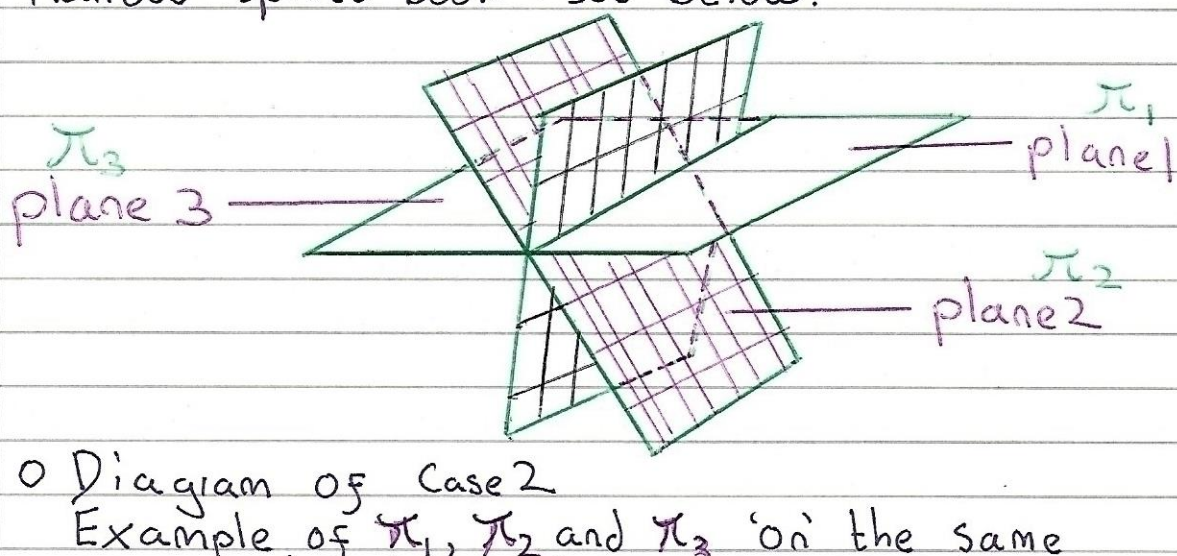
We shall consider these one at a time.
Let us call the three planes π_1, π_2 and π_3.
Now let n_1, n_2, n_3 be vectors normal to these planes respectively.

Case 1

The three planes are coincident.
Then every point on each plane lies on all three planes. So the intersection of the planes is a plane, which is π_1 $(=\pi_2=\pi_3)$.

Case 2

If π_1, π_2 and π_3 all contain the same line but are not all three identical, then the intersection of the planes is that line.
This can happen either when two of the planes coincide and the third is not parallel to these two. Or when the normals of the planes are all distinct, but all parallel to a given plane.
In this last case the planes fan out from the line of intersection. Like pages from a narrow spined book - see below.



○ Diagram of Case 2
Example of π_1, π_2 and π_3 'on' the same line, but not all three are identical.

Orthogonal Projections onto a Plane — Vectors

Summary

① The scalar product of the vectors 'a' and 'b' is defined as $a \cdot b = |a|\,|b| \cos\theta$, where θ is the angle between the directions of 'a' and 'b'. Or if $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$. $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$.

② If $a.b = 0$ then $a = 0$ or $b = 0$, or a and b are orthogonal.

③ The length of a vector is connected with the scalar product as: $|a|^2 = a.a$.

④ The cosine of the angle θ between the directions of 'a' and 'b' (as defined in point ①) is given by:

$$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 + b_3}{\sqrt{a_1^2 a_2^2 a_3^2}\,\sqrt{b_1^2 b_2^2 b_3^2}}$$

⑤ The component of 'a' in the direction of 'b' is $a.\hat{b}$.

⑥ $a.b$ is the product of the components of the two vectors in the direction of either.

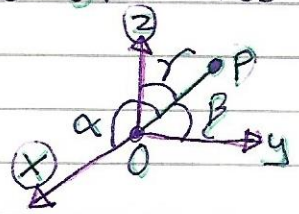
⑦ The equation of a plane is $(r-a).n = 0$, where 'n' is a vector orthogonal to the plane, and 'a' is the position vector of a point on the plane. If $n = n_1 i + n_2 j + n_3 k$ and $a = a_1 i + a_2 j + a_3 k$, the cartesian equation of the plane is $n_1 x + n_2 y + n_3 z = n_1 a_1 + n_2 a_2 + n_3 a_3$

The intersection is the whole plane 'coincident'

⑧ Two planes intersect on a straight line unless (1) they are parallel and do not intersect at all, or ② they are coincident.

⑨ The intersection of 3 planes can be a plane, line or a point. Or the empty set (where no point lies on all 3 planes)

3 Dimensional Space - Equations etc.. Vectors

• Distance	The distance $\lvert P \rvert$ of the point $P = (x, y, z)$ from the origin O is given by $$\lvert P\rvert = \sqrt{x^2 + y^2 + z^2}$$
• Two Points Distance	which is an extension of Pythagora's theorem. $\lvert P\rvert$ is called the 'modulus' of P. By further extension the distance of point A (a_1, a_2, a_3) and B (b_1, b_2, b_3): $$\lvert A - B\rvert = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$$
• Modulus	The points with unit modulus all lie on the unit sphere S^2. Any non-zero point P can be converted into one with unit modulus by $$\hat{P} = \frac{P}{\lvert P\rvert}, \quad P \neq 0$$
• ^ Direction of cosines	$\hat{P}$ is the 'direction of cosines' of P. If $\hat{P} = (\cos\alpha, \cos\beta, \cos\gamma)$ then $\cos\alpha^2 + \cos\beta^2 + \cos\gamma^2 = 1$ and the angles α, β, γ are the angles which the line OP makes with the coordinate axes...
• Dot Product	 • The scalar or dot product generates a real number. $= A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$ • The dot product satisfies these: $A\cdot A = \lvert A\rvert^2$, $A\cdot B = B \cdot A$, $(cA)\cdot B = A\cdot(cB) = c(A\cdot B)$ $(A+B)\cdot C = A\cdot C + B \cdot C$.
• Coordinate Free	The intrinsic, coordinate-free definition: $A \cdot B = \lvert A\rvert \, \lvert B\rvert \cos\theta$ where θ is the angle $\angle AOB$. So A, B are orthogonal if $A\cdot B = 0$.
• Crossproduct	The crossproduct takes two vectors and supplies a third by the rule $A\times B = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$. Unlike the dot produt, the crossproduct is anti-symmetric, that is $A\times B = -B\times A$. $A\times A = 0$, so A is parallel to B only if $A\times B = 0$.
• Scalar Triple	Scalar Triple product. If we combine the cross and dot products we get the triple scalar, or scalar triple product $(A\times B)\cdot C = a_2b_3c_1 - a_3b_2c_1 + a_3b_2c_1 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$ The notation $[ABC]$ is used for $(A\times B)\cdot C$
• Isometry	An isometry is a transformation that maintains/preserves distance. An example is $X \to T(x)$ a transformation of space that T is an isometry if $\lvert T(x) - T(y)\rvert = \lvert x - y\rvert$ for all x, y.

Projections – Orthogonal [Vectors]

- Orthogonal Projections – If ϖ is a plane in space, say $A \cdot P = d$, then we can define an orthogonal projection of P from R^3 to the plane ϖ by sending a point X from R^3 to the nearest point on ϖ.

Point Projecting

Projecting a point by the formula
$$p(x) = X + \frac{d - A \cdot X}{|A|^2} A$$

Finding the orthogonal projection of the line
$$\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-4}{2}$$
in the plane $2x - y + z + 3 = 0$

Image Plane

An orthogonal projection is determined by the image plane and a direction in the plane which gives the first coordinate. The second direction is automatically defined by 'the right handedness condition'.

Right Handed Basis

An ordered basis is called right handed if a rotation from A to B causes a right-handed screw to move in the positive direction along C. Otherwise it's left-handed.

Plane of projection

Let the plane of projection be $A \cdot P = 0$ and let B be a point in the plane chosen so that the line OB is the first coordinate axis or x-axis of the target. Then A, B are orthogonal and we can always assume they have unit length.

A suitable candidate for the second or y-axis in the target plane is $A \times B$. Since $\{A, B, A \times B\}$ is a right-handed basis for R^3 every point $X \in R^3$ can be written uniquely as $X = xB + yA \times B + zA$. The projection takes this to $p(X) = xB + yA \times B$. The values of x, y can be determined by dot products and so
$$x = X \cdot B, \quad y = X \cdot (A \times B)$$

Solution

Hence the corresponding orthogonal projection R^3 to R^2 is
$$p(X) = (X \cdot B, [XAB])$$

Another Example

Example Orthogonally project the cube with vertices $(\pm 1, \pm 1, \pm 1)$ onto the plane $x + 2y + 3z = 0$ with x-axis along $(1, 1, -1)$. Then $A = (1, 2, 3)/\sqrt{14}$, $B = (1, 1, -1)/\sqrt{3}$, and $A \times B = (-5, 4, -1)/\sqrt{42}$. So
$$P(x, y, z) = ((x + y - z)/\sqrt{3}, (-5x + 4y - z)/\sqrt{14}$$
Cube vertices become $((\pm 1 \pm 1 \mp 1)/\sqrt{3}, (\mp 5 \pm 4 \mp 1/\sqrt{14})$

[Hyperplanes] [Vectors]

A hyperplane in R^n is a collection of points, $X=(x_1, x_2, \dots, x_n)$ which satisfy the linear equation of the form

$$a_1x_1 + a_2x_2 \dots a_nx_n = b$$

n=2 is a line

If $n=2$ a hyperplane is a line and so the general equation of a line in the plane is $ax+by=c$ where the numbers a, b are not zero

Through two points in R^n there is a unique line. Let $P=(-1,5)$, and $Q=(0,3)$ be the two points in R^2 and let l be the unique line through P and Q. All points (x,y) on l are given by the equation $2x+y=3$.

Cramer Line Equation

We can find the equation of the line l by the method due to Cramer. Suppose that $P=(x_1, y_1)$ and $Q=(x_2, y_2)$ writing the coordinates of the points in an array:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

Array

The coefficients of the equation of the line are given by the three 2×2 determinants with alternating sign by succesively deleting the coloms of the array. The above array is

where $P=(-1,5)$ and $Q=(0,3)$ $\begin{vmatrix} -1 & 5 & 1 \\ 0 & 3 & 1 \end{vmatrix}$ $\begin{vmatrix} P(-1) & P(5) & 1 \\ Q(0) & Q(3) & 1 \end{vmatrix}$

and the determinants with sign are

$$\begin{vmatrix} 5 & 1 \\ 3 & 1 \end{vmatrix} = 2, \quad -\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} -1 & 5 \\ 0 & 3 \end{vmatrix} = -3$$

Solution

So the equation of the line is $2x+y-3=0$

If you are curious as to why this works, note that if a point (x,y) lies on the line defined by the two points $(x_1,y_1), (x_2,y_2)$ then the area of the resulting triangle is zero. So the formula above for the area of a triangle we have

Area of triangle zero for line with point lying on it.

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

The equation of the line is obtained by expanding this determinant using the top row.

[Stereographic Projections]

Vectors

Depth to orthographic projections

To better mimic what the eye sees, we can add the perspective of depth, to otherwise 'purely' 2D orthographic projections. The result is that objects that are nearer appear larger, and those further away - smaller.

Projection Point

Let P be the eye's pupil and the projection point. If X is a point in $\mathbb{R}^3$ let X' be the point where the line XP meets the plane, ϖ.

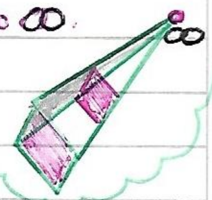
To fix coordinates let A, B be orthogonal and of unit length and let ϖ be the plane passing through O and orthogonal to A.

Add depth

So B and $A \times B$ lie in ϖ.
Let $X' = tX + (1-t)P$. Then
$0 = X' \cdot A = tX \cdot A + (1-t)P \cdot A$.
So $t = -P \cdot A / (X-P) \cdot A$
and

$$X' = -\frac{P \cdot A}{(X-P) \cdot A} X + \frac{X \cdot A}{(X-P) \cdot A} P$$

- The projection is not defined on points on the plane $X \cdot A = P \cdot A$. These points can be thought as going to ∞.



Solution

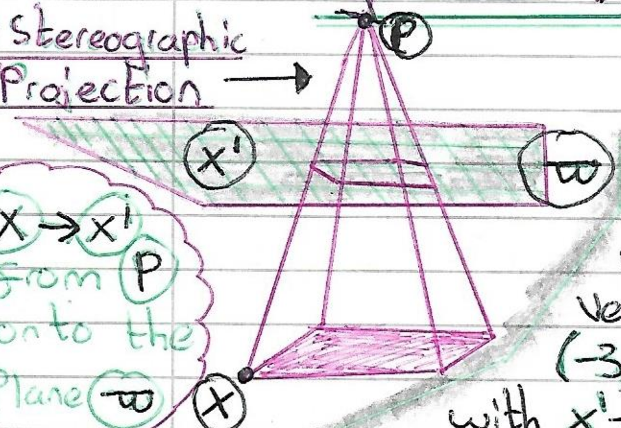
We now want to identify X' with a point of $\mathbb{R}^2$.
If $X' = x'B + y'A \times B$ then

$$x' = X' \cdot B = \frac{(X \times P) \cdot (A \times B)}{(X-P) \cdot A}$$

$$y' = X' \cdot A \times B = -\frac{(X \times P) \cdot B}{(X-P) \cdot A}$$

These are the coordinates of the stereographic projection shown below

- Stereographic Projection →



$X \to x'$ from P onto the Plane ϖ

Example

Example: Suppose we want to project stereographically the cube with vertices $(\pm 1, \pm 1, \pm 1)$ using the pupil $(-3, -6, -9)$ onto the plane $x + 2y + 3z = 0$ with x'-axis along $(1, 1, -1)$. Then with the notation above $A = (1,2,3)/\sqrt{14}$, $B = (1,1,-1)/\sqrt{3}$, and $A \times B = (-5, 4, -1)/\sqrt{42}$. The coordinates of stereographic projection are therefore

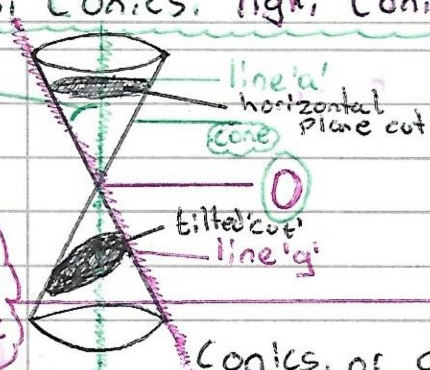
$$x' = -\frac{\sqrt{42}\,(x+y-z)}{x+2y+3z+3\sqrt{14}} \qquad y' = -\frac{\sqrt{3}\,(-5x+4y-z)}{x+2y+3z+3\sqrt{14}}$$

(The coordinates of the cube vertices are now obtained by substitution)

Conics! Conics! Agh! Conic Sections!

Vectors

α (the semi angle)



General equation of a conic

- Two equations for a conic
$x^2 + y^2 = z^2 \tan^2 \alpha$ which is from $ax^2 + 2hxy + by^2$ becoming $\lambda X^2 + \mu Y^2$ from the general form
$ax^2 + 2hxy + by^2 + 2cx + 2dy + k = 0$

Conics, or conic sections, are curves which are the intersection of a complete cone in R^3.

Forming the conic

Let a line 'g' meet another 'a' in R^3, so they connect at point 'O'. Let the line 'g' rotate around line 'a', at an angle of α between the two lines at point 'O'.

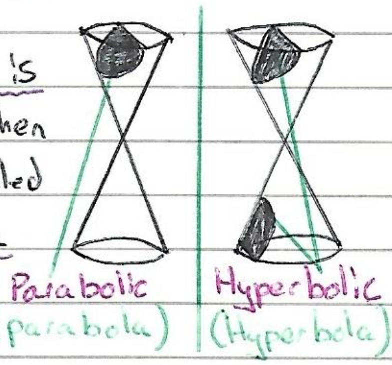
Rotating line 'g' (or the generator) about line 'a' (or the axis), traces out a cone.

Axis is tilted

- If line 'a' is vertical and not slanted the 'cut' at the top will be circular, if the axis line is slanted the 'cut' will be an elipse.

Hyperbola Parabola

If the plane is further tilted so that it is parallel to a generator of the cone then the plane cuts the cone in a curve called a parabola. Further tilting results in a plane meeting the cone in two parts and cuts the cone in a hyperbola.



If the plane passes through O, the apex of the cone, then the intersection is called a degenerate conic. These are a point (the apex) or a pair of lines through O (two generators), or a line (one generator where the plane is tangent).

Elipse

The circle can be regarded as a special case of an elipse in many cases. We look at elipses here in regard to the conic. An elipse here has $a > b$ (see diagram). (next page)

Algebraic Equation

The algebraic equation for an elipse in canonical position is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$ This curve meets the coordinate axes at $(\pm a, 0), (0, \pm b)$

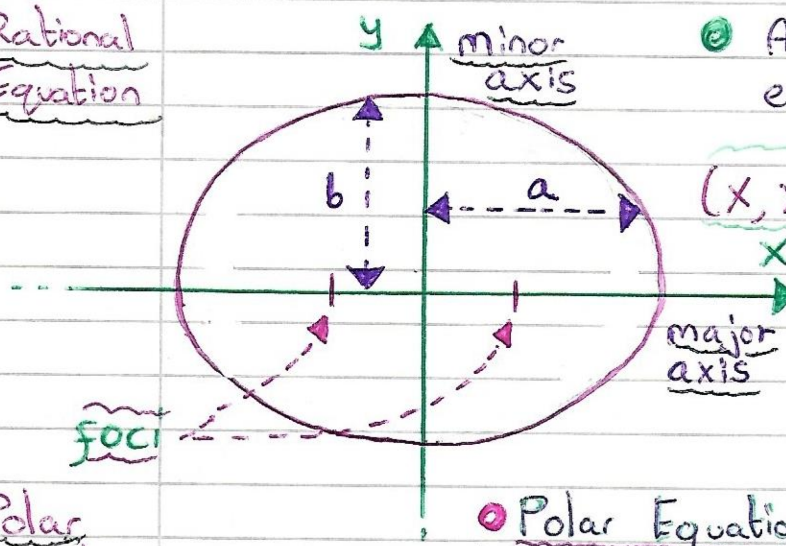
Parametric Equation

The standard parametric equation of the ellipse in canonical position is: $r = (x, y) = (a \cos \varphi, b \sin \varphi)$ $(0 \leq \varphi \leq 2\pi)$ $(a > b > 0)$
In this case φ is not the polar angle θ. It is the polar angle of the point Q having coordinate $(x, \pm\sqrt{a^2 - x^2})$ on the circle $x^2 + y^2 = a^2$ (the auxiliary circle). Point Q is vertically above/below point P having coordinates (x, y) on the elipse.

Elipses and Hyperbolas

(Vectors)

Rational Equation



- A rational parametric equation for an elipse

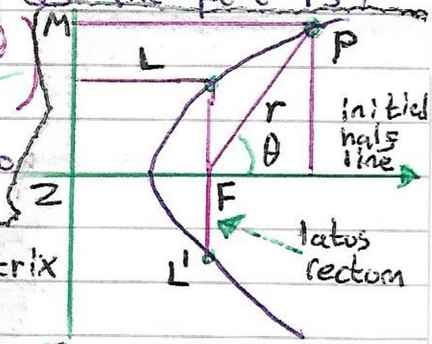
$$(X,Y)=\left(\frac{A(B^2-t^2A^2)}{B^2+t^2A^2},\ \frac{2tAB^2}{B^2+t^2A^2}\right)$$

which gives a rational parametrisation of the elipse in the XY-plane.

Polar Equation

- Polar Equation for an elipse can be found by substituting $x=r\cos\theta$ and $y=r\sin\theta$ into the algebraic equation $\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$ $(0\le\theta\le 2\pi)$ $(a>b>0)$

- In this equation the center of the elipse is below the pole, and the positive x-axis is the initial half-line. The polar equation with a focus as the pole is: $\frac{l}{r}=1-e\cos\theta$ $\left(\text{or } \frac{l}{r}=1+e\cos\theta\right)$



(and has the initial line perpendicular to the latus rectum). The latus rectum of a conic is the chord LFL' (directrix) through the focus F which is bisected at the focus, or the line that has this chord. The semi-latus rectum l is one half of the length of the chord LFL'.

Polar Equation of a conic with focus as a pole

Theorem Semi Latus Return of elipse hyperbola parabola

Theorem - The Semi-Latus return of...

- the elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(b<a)$ is $l=\frac{b^2}{a}$
- the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $l=\frac{b^2}{a}$
- the parabola $y^2=4ax$ is $l=2a$

Proof - For $e\neq 1$ we have succesfully (semi-latus return) → $\frac{l}{r}=1-e\cos\theta$

$$l=r-er\cos\theta$$

$$(l+ex)^2=r^2=x^2+y^2$$

$$x^2(1-e^2)+y^2-2elx=l^2$$

$$(1-e^2)\left(x-\frac{el}{1-e^2}\right)^2+y^2=\frac{l^2}{1-e^2}$$

(and)

$$(1-e^2)X^2+y^2=\frac{l^2}{1-e^2}$$

where $X=x-\frac{el}{1-e^2}$

Hyperbolas and Parabolas

Vectors

Theorem Semi Latus Return of elipse hyperbola parabola Proof Continued

Proof of semi-latus return - continued

In the case previous we proved for $e \neq 1$, for the case where $e < 1$ our proof is →

$$\frac{X^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a^2 = \frac{l^2}{(1-e^2)^2} > \frac{l^2}{1-e^2} = b^2$$

Therefore for $e<1$ we have $\frac{b^2}{a^2} = 1 - e^2$ and $\frac{b^2}{a} = l$

Memoire

- Unique Solution Central Conic
- Many Solutions Parallel Lines
- No Solution Parabola

- For $e>1$ $\frac{X^2}{a^2} - \frac{y^2}{b^2} = 1$

where

$$a^2 = \frac{l^2}{(e^2-1)^2} \text{ and } b^2 = \frac{l^2}{e^2-1}$$

Therefore $\frac{b^2}{a^2} = e^2 - 1$ and $\frac{b^2}{a} = l$

- For $e=1$ we have seen above that

$y^2 = 2lx + l^2 = 2l\left(x + \frac{1}{2}l\right) = 2lX$ where $X = x + \frac{1}{2}l$

which is the parabola $y^2 = 4aX$ with $l = 2a$

Conics with Centres

- A Central Conic is a conic with a centre of symmetry. A centre is a point such that all chords through the centre are bisected at the centre. So a general conic with the previous eq.

$$ax^2 + 2hxy + by^2 + 2cx + 2dy + K = 0$$

has a centre at the origin if $(-x, -y)$ lies on the conic whenever (x, y) lies on the conic. It only happens when $c = d = 0$.

- Parabolas are not central. Ellipses, hyperbolae, and pairs of distinct intersecting lines will have unique centres. Parallel or iterated lines will have a line of centres

Centre Of A Conic

- The Centre Of A Conic - Proof with the two linear equations

$ax + hy + c = 0$ and $hx + by + d = 0$

The partial derivatives are

$\varphi(x,y) = ax^2 + 2hxy + by^2 + 2cx + 2dy + K$

And the centre of $\varphi(x,y) = 0$ given by

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial y} = 0$$

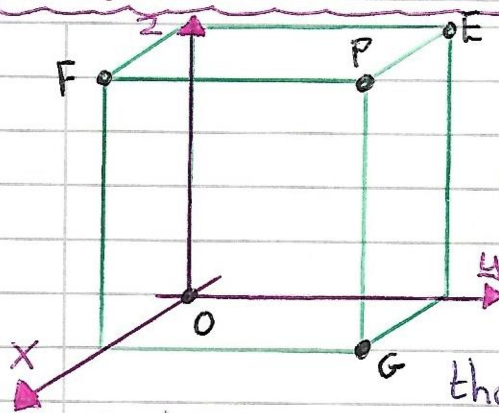
Case 1 - no solution (parabola)

Case 2 - one solution (unique centre, or $\varphi(x,y)=0$ has one or none solutions.

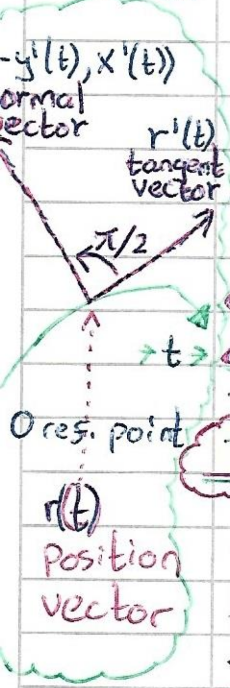
Case 3 - Many solutions, parallel or iterated lines, $\varphi(x,y)=0$ has no solution

3 Dimensional Review

Vectors



Suppose $P=(x,y,z)$ is a general point. Let $E=(0,y,z)$, $F=(x,0,z)$, $G=(x,y,0)$ be the nearest points of the coordinate planes to P, and let $X=(x,0,0)$, $Y=(0,y,0)$, $Z=(0,0,z)$ be the nearest points of the coordinate axes. Then O, X, G, Y, Z, F, P, E form the vertices of a retangular box.



Points may be added $A+B=(a_1\, a_2\, a_3)+(b_1\, b_2\, b_3)=(a_1+b_1, a_2+b_2, a_3+b_3)$, or multiplied by a real number $cA=c(a_1,a_2,a_3)=(ca_1, ca_2, ca_3)$. We will denote by $I=(1,0,0)$, $J=(0,1,0)$ and $K=(0,0,1)$ the unit points on the coordinate axis.

Since every point $P=(x,y,z)$ can be written uniquely as $P=xI+yJ+zK$, it follows that $\{I,J,K\}$ is a basis for $\mathbb{R}^3$. Any weighted sum such as $P=xI+yJ+zK$ is called a linear combination of $\{I,J,K\}$.

The Vector Triple Product

Unlike ordinary multiplication, the cross product is not in general associative. That is $A\times(B\times C)\neq(A\times B)\times C$. However there is a formula that enables us to identify both sides -

The triple vector product: $(A\times B)\times C=(A\cdot C)B-(B\cdot C)A$

To prove this we could expand both sides using various formulea and check the results were equal.

- Or, we assume that A, B are not parallel. Now $X=(A\times B)\times A$, is orthogonal to A and lies on the plane AB. Also by right-handedness B and X lie on the same side of the plane containing A and $A\times B$. Moreover $|X|$ is $|A|^2\,|B|\sin\theta$ where θ is the angle $\angle AOB$.
- Then $(A\cdot A)B-(A\cdot B)A$ satisfies these requirements that uniquely define X and so:

$$X=(A\times B)\times A=(A\cdot A)B-(A\cdot B)A$$

Similary $(A\times B)\times B=(A\cdot B)B-(B\cdot B)A$

Now $\{A, B, A\times B\}$ is a basis for $\mathbb{R}^3$ and so for some x,y,z we have $C=xA+yB+zA\times B$

- Expanding $(A\times B)\times(xA+yB+zA\times B)$ and using the fact $A\times B$ is orthogonal to A and B we get:

$$(A\times B)\times C=x(A\cdot A)B-x(A\cdot B)A+y(A\cdot B)B-y(B\cdot B)A$$
$$=(-xA-yB-z(A\times B))\cdot BA+(xA+yB+z(A\times B))\cdot AB$$
$$=(A\cdot C)B-(B\cdot C)A.$$

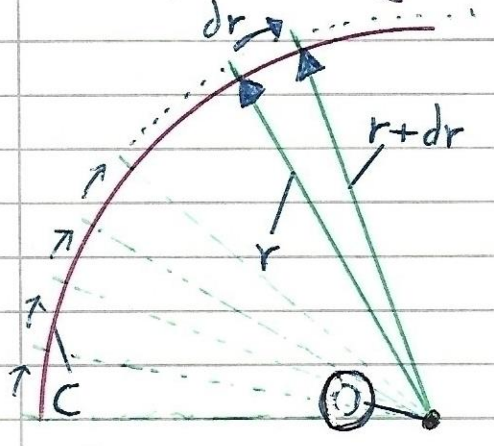
This proves the three-dimensional nature. The vector product in $\mathbb{R}^7$ has no such expansion.

Line Integrals

Vectors

Consider a particle moving along a curved path **C** through space. The particle is acted on by Force **F(r)** - which is a vector field. Finding the total amount work done as the particle moves along curve C.

First we divide the curve into a large number of small segments, or pieces. The find the work done as the particle moves through each of these segments from position **r** to **r** + d**c**.



The work done is $-\mathbf{F} \cdot d\mathbf{r}$.
The total amount of work done is therefore,

$$W = \sum_{i=1}^{N} -\mathbf{F}_i \cdot d\mathbf{r}_i$$ ...the sum of all contributions along the curve.

The Line integral of F along the curve C is defined by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{N \to \infty} \sum_{i=1}^{N} \mathbf{F}_i \cdot d\mathbf{r}_i$$

- the vector **dr** is often reffered to as a **line element**.

Note that the direction of the integral along curve C must be specified. So, if the direction of the curve is reversed, all elements dr are reversed also and the integral is multiplied by −1.

Evaluating

Evaluating Line Intregrals Line intregrals are evaluated using another parameter, such as t. So in the above:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt.$$ Giving the value of position vector **r** in terms of t.

This can be regarded as integration by substitution.

Vector Field

Suppose curve C is given in terms of t by $x = t$, $y = t$, $z = 2t^2$, and t is in the range $0 \le t \le 1$. Then as t moves between 0 and 1, the position vector $\mathbf{r} = (x, y, z)$ traces along curve C, connecting the points $(0,0,0)$ to $(1,1,2)$. With a vector field **F** being defined as $\mathbf{F} = (y, x, z)$. To evaluate the line intregral both F, and $d\mathbf{r}/dt$ need to be written in terms of t so substituting $x = t, y = t, z = 2t^2$ for F.. gives $\mathbf{F} = (t, t, 2t^2)$ and...

$$\frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (1, 1, 4t)$$

Line Integrals - continued

Vectors

(continued) The Line Integral can now be evaluated

$$\int_C F\cdot dr = \int_0^1 (t, t, 2t^2)\cdot(1, 1, 4t)\,dt = \int_0^1 2t + 8t^3\,dt = 3$$

In this first example the parmetric form of the curve C was given. If the curve C is given in a different form, then a pareametric form must be constructed so that the integral can be evaluated.

For example suppose now that $F=(y,x,z)$ as before, but C is the straight line connecting the origin to the point $(1,2,3)$.

The way in which curve C is parametrised is not unique, so we can make the arbitary choice $x=t$.

Since x varies between 0 and 1 along the line, this is also the range for t.

The end point of C is $(1,2,3)$, so y and z must be given by $y=2t$, $z=3t$, and so $dr=(1,2,3)\,dt$.

The value of the integral is therefore

$$\int_C F\cdot dr = \int_0^1 (2t, t, 3t)\cdot(1,2,3)\,dt = \int_0^1 13t\,dt = 6.5$$

A simple closed curve is a curve that does not intersect itself. Anywhe

Such an integral - $\oint A\cdot dr = \oint A_1dx + A_2dy + A_3dz$. In aerodynamics and fluid mech. this is called the circulation of A about C, where A is velocity of fluid.

Line integrals sometimes occur over curves that are closed. Ie- when the starting point and end point of the curve are equal. In this case the integral is written using the symbol $\oint$, which indicates the integral is along a closed curve.

- Consider the integral of $F=(y,x,z)$ around the closed curve given by $x=\cos\theta$, $y=\sin\theta$, $z=0$. Where $0\leq\theta\leq 2\pi$. Here as θ varies, the curve C describes a circle in the x,y plane

The line element dr is expressed in terms of the parameter θ as $dr=(dx,dy,dz)=(-\sin\theta,\cos\theta,0)\,d\theta$

So the value of the line integral is

$$\oint_C F\cdot dr = \int_0^{2\pi}(\sin\theta,\cos\theta,0)\cdot(-\sin\theta,\cos\theta,0)\,d\theta$$

$$= \int_0^{2\pi} -\sin^2\theta + \cos^2\theta\,d\theta = \int_0^{2\pi}\cos 2\theta\,d\theta = \left[1/2\sin 2\theta\right]_0^{2\pi} = 0$$

The line integral of a vector field F, around a closed curve is often called the circulation of F around C

Integration Methods

Vectors

A rod of length 'a' has a mass per unit length $p(x)$ that varies along the length of the rod according to the formula $p(x) = 1+x$. Find the rods total mass.
Consider dividing the rod into N small sections, each of length dx_i. The mass of each section is $p(x_i)dx_i$. The total mass M of the rod is the sum of the masses of all the $p(x_i)\,dx_i$ sections.

$$M = \sum_{i=1}^{N} p(x_i)\,dx_i$$

The integral of $p(x)$ is defined to be the limit of this sum as $N \to \infty$.

$$\int_0^a p(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} p(x_i)\,dx_i$$

The total mass M is therefore

$$M = \int_0^a 1+x\,dx = \left[x + x^2/2\right]_0^a = a + a^2/2$$

Speed over time

Further Example A vehicle starts from rest and accelerates uniformly up to a speed of 10 m/s over a time of 20s. What is the total distance travelled in this time?
It's speed at time t is $v(t) = t/2$ m/s. In a small time interval dt the distance travelled is $v(t)=dt = t/2\,dt$. The total distance S travelled in the total time of 20s is therefore

$$S = \int_0^{20} t/2\,dt = \left[t^2/4\right]_0^{20} = 100\text{m}$$

Substitution

Substitution Example In this method a complicated integral is transformed to a simpler on by a change of variable. The choice may not be easy to find the most sensible substitution in all cases. ~ Evaluate $\int x/\sqrt{1-x}\,dx$
≈ Here, the difficulty is caused by the $\sqrt{1-x}$ in the denominator. This suggests that the appropiate substitution is $u = 1-x$, so $x = 1-u$ and $dx = -du$. The integral becomes

$$\int -(1-u)/\sqrt{u}\,du = \int -1/\sqrt{u} + \sqrt{u}\,du = -2u^{1/2} + 2u^{3/2}/3 + c$$

where c is an arbitrary constant of integration. The result in terms of x is $-2\sqrt{1-x}\,(2+x)/3 + c$

Substitution $\int\sqrt{1-x^2}\,dx$

Continuing examples of Integration Substitution, evaluate

$$\int\sqrt{1-x^2}\,dx$$

For integrals involving the quantity $\sqrt{1-x^2}$, the appropriate substitution is $x=\sin\theta$ (or $x=\cos\theta$, which would do equally well). With this choice, $\sqrt{1-x^2}$ becomes $\cos\theta$ and $dx=\cos\theta\,d\theta$. The integral simplifies to

$$\int\cos^2\theta\,d\theta.$$

Integrals of this type, which occur very frequently, are evaluated using the trigonometric formula $\cos^2\theta=(1+\cos 2\theta)/2$, so the value of the integral is $(2\theta+\sin 2\theta)/4+c$. In terms of x this result can be written

$$\int\sqrt{1-x^2}\,dx=\left(\sin^{-1}x+x\sqrt{1-x^2}\right)/2+c$$

Substitution $\int_0^1 x^2\sqrt{1-x^2}\,dx$

As a final example of substitution evaluate

$$\int_0^1 x^2\sqrt{1-x^2}\,dx$$

Again the substitution $x=\sin\theta$ is used. Since the limits of integration are given, these can also be expressed in the new variable θ. When $x=0$, $\theta=0$, and when $x=1$, $\theta=\pi/2$. So the integral becomes

$$\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

This can be simplified to

$$\int_0^{\pi/2}1/4\sin^2 2\theta\,d\theta=\int_0^{\pi/2}1/8(1-\cos 4\theta)\,d\theta=\pi/16$$

Integration by parts

Integration By Parts Integration by parts is an important and useful technique used when an integral involves a product of two terms. The integration by parts formula is derived from the product rule for differentiation. Given two functions of x, $u(x)$ and $v(x)$, the rule for the derivative of their product is

$$\frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$

Integrating this expression and rearranging the terms gives the integration by parts formula.

$$\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$$

As with the case of integration by substitution, some experience is helpful in determining whether this formula would be useful in evaluating an integral.

Integration Methods

Vectors

Examples Integration by parts

(integration by parts - continued)... Experience is also helpful in finding a way to split the integral into the two parts.

In general, it is best to choose u to be a function which becomes simpler when differentiated. The following two examples illustrate the use of the method of integration by parts

Evalute $\int x \sin x \, dx$ In this example we choose $u = x$, $dv/dx = \sin x$, so $v = -\cos x$. Applying the formula $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ gives

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$$

Note that it is essential to make the right choice for u and v. If we had chosen $u = \sin x$, dv/dx then the resulting int ral would have involved $x^2 \cos x$ which is more complicated than the integral we started with.

Example

Final Example - Integration by parts

Evaluate...

$$\int \exp ax \cos x \, dx$$

For this case two applications of $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ ✱ are needed. Choosing $u = \exp ax$, $dv/dx = \cos x$..

$$\int \exp ax \cos x \, dx = \exp ax \sin x - \int a \exp ax \sin x \, dx + c$$

Appling the formula ✱ to the integral on the rhs with $u = a \exp ax$, $dv/dx = \sin x$.

$$\int \exp ax \cos x \, dx = \exp ax \sin x + a \exp ax \cos x - \int a^2 \exp ax \cos x \, dx + c$$

At this stage it may seem no progress has been made, since the original integral is back on the right side. But by rearranging the terms - $(1+a^2) \int \exp ax \cos x \, dx = \int \exp ax \sin x + a \exp ax \cos x + c$ and so the value of the original integral is $\int \exp ax \cos x \, dx$

$$= (\exp ax \sin x + a \exp ax \cos x + c)/(1+a^2)$$

① Here the choice of u or v doesn't matter. The result can be got with $u = \cos x$, $dv/dx = \exp ax$. Provided a similar choice for u to be the trig. term, and dv/dx to be the exponential term.

Modelling · The Heat Equation · Vectors

Consider a uniform rod of length 1 with initial temperature u of the entire rod equal to 1. Then, at $t=0$, we start cooling the rod at the endpoints $x=0$ and $x=1$. By an appropriate choice of scales, the heat equation below models the temperature distribution with $f(x)=1$ in the rod for $t>0$.

$$u_t = u_{xx} \quad \text{for } x \in (0,1),\ t>0$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = f(x)$$

In order to find the temperature, we have to represent the function $f(x)=1$ as a finite sum of sine functions. However this is impossible and the procedure fails at the simplest possible initial condition. On the other hand, if we allow infinite linear combinations, it can be shown that

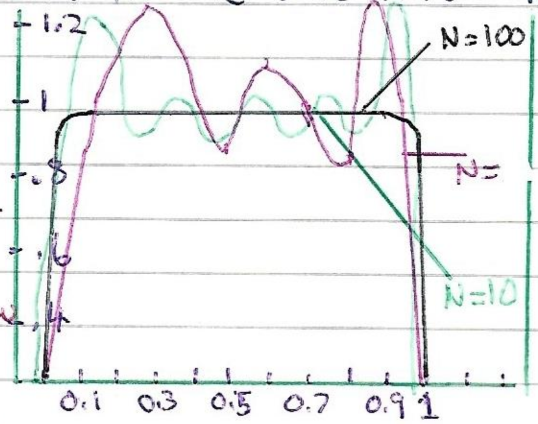
$$1 = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin((2k-1)\pi x)$$

for x in the unit interval.

Here initial function f is written as a finite linear combination of eigenfunctions $\{\sin(k\pi x)\}$. So there are constants $\{c_k\}_{k=1}^{N}$ so that $f(x)=\sum_{k=1}^{N} c_k \sin(k\pi x)$

The simplest function is represented and expressed by an infinite series.

From the plot of $N=3, 10$, and 100, it's easy to see that the series converges towards $f(x)=1$ within the unit interval. Notice that the covergence is very slow near the boundaries.



When allowing infinite series in the initial data, the solution is given by →

$$u(x,t) = \sum_{k=1}^{N} c_k e^{-(k\pi)^2 t}\sin(k\pi x)$$

also becomes an infinite series. From taking this equation we get the following formal solution.

$$u(x,t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}e^{-((2k-1)\pi)^2 t}\sin((2k-1)\pi x)$$

This is called a 'formal solution' because we have not yet proved the series and its derivatives converge and satisfy all the requirements of the heat equation (see margin).

From this example it's important to see that finite linear combinations of eigen-functions don't cover all interesting initial functions of $f(x)$. So we are led to allow infinite linear combinations of the form

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

By letting N tend to infinity we get

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t}\sin(k\pi x)$$

First-Order Homogeneous Equations

Vectors

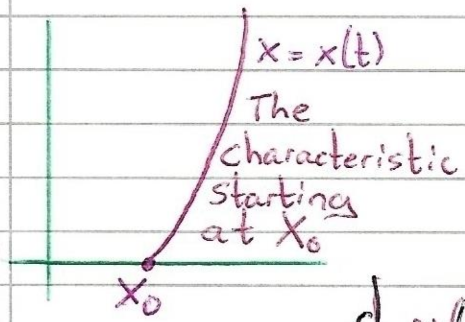
A smooth function is continuously differentiable as many times as we find neccessary. Not all functions can be that smooth but is a good first 'guess' when describing smooth functions.

Consider the following first-order homogeneous partial differential equation $u_t(x,t) + a(x,t)u_x(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$ with the initial condition $u(x,0) = \phi(x)$, $x \in \mathbb{R}$.
Here we assume the variable coefficient $a = a(x,t)$ and the initial condition $\phi = \phi(x)$ to be given smooth functions.
A problem in the form of the equations above is reffered to as a Cauchy problem. We usually refer to t as the time variable and x as the spatial coordinate.
We want to derive a solution using the method of characteristics. Those characteristics of the equations above are curves in the x-t plane defined as follows:

For a given $x_0 \in \mathbb{R}$, consider the ordinary differential equation →

$$\frac{dx(t)}{dt} = a(x(t),t), \quad t > 0$$
$$x(0) = x_0$$

The solution $x = x(t)$ of this problem defines a curve $\{(x(t),t), t \geq 0\}$ starting in $(x_0, 0)$ at $t = 0$.



Now we want to consider u along the characteristic, ie. we want to study the evolution of $u(x(t),t)$. By differentiating u with respect to t we get

The initial condition here is the Initial Boundary Value Problem.

$$\frac{d}{dt}u(x(t),t) = u_t + u_x \frac{dx(t)}{dt}$$
$$= u_t + a(x,t)u_x = 0$$

Where we have used the definition of $x(t)$ given by

$$\frac{dx(t)}{dt} = a(x(t),t), \ t > 0, \ x(0) = x_0$$

and $u_t(x,t) + a(x,t)u_x(x,t) = 0$, $x \in \mathbb{R}$, $t > 0$.

Since $\frac{d}{dt}u(x(t),t) = 0$ the solution is constant along the characteristic.

So $u(x(t),t) = u(x_0, 0)$ or $u(x(t),t) = \phi(x_0)$.
This means that if, for a given $a = a(x,t)$, we are able to solve the ODE $\frac{dx(t)}{dt} = a(x(t),t)$, $t > 0$, $x(0) = x_0$, we can compute the solution of the Cauchy problem.

Matrices Review

Vectors

A matrix can be complex; it's entries are complex numbers. For a matrix A, its complex conjugate A^* is found by taking the complex conjugate of each of its elements.

For the Hermitian conjugate $A^\dagger$, we take the transpose of its complex conjugate matrix.

Orthogonal matrices are such if $A^{-1} = A^T$. Consider the rotation matrix of $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ it's inverse and transpose are

$$A^{-1} \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Since $\cos^2\theta + \sin^2\theta = 1$, we have $A^T = A^{-1}$. Therefore the original rotation matrix is orthogonal.

A useful concept in computational mathematics is quadratic forms. For a real vector $q^T = (q_1, q_2, q_3 \ldots q_n)$ and a real symmetric square matrix A, a quadratic form $\psi(q)$ is a scalar function defined by

$$\psi(q) = q^T A q = (q_1\, q_2 \ldots q_n) \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

which can be written as

$$\psi(q) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_i A_{ij} q_j$$

Since ψ is a scalar, it should be independent of the coordinates. In the case of the square matrix A, ψ might be more easily evaluated in certain intrinsic coordinates $Q_1, Q_2, \ldots Q_n$. An important result concerning the quadratic form is that it can always be written through appropriate transformations as

$$\psi(q) = \sum_{i=1}^{n} \lambda_i Q_i^2 = \lambda_1 Q_1^2 + \lambda_2 Q_2^2 + \ldots \lambda_n Q_n^2$$

where λ_i are the eigenvalues of the matrix A determined by $\det|A - \lambda I| = 0$, and Q_i are the intrinsic components along directions of the eigenvectors in this case.

Eigenvalues and Eigenvectors The eigenvalues λ of any $n \times n$ square matrix A are determined by $Au = \lambda u$, or $(A - \lambda I)u = 0$, where I is a unitary matrix with the same size as A. Any non-trivial solution requires that $\det|A - \lambda I| = 0$ or $\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$ which can be written as the polynomial $\lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0 = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n) = 0$ where λ_i are the eigenvalues which could be complex numbers. In general the determinant is zero, which leads to a polynomial of order n in λ. For each eigenvalue, there is a unique eigenvector, however lengths are not unique.

Integrals Review

Vectors

Power/Algebraic Functions

If $P_1(x)$ and $Q_1(x)$ are arbitary polynomials, their ratio can be represented in the form $\frac{P_1(x)}{Q_1(x)} = P_2(x) + \frac{P(x)}{Q(x)}$ where $P_2(x), P(x), Q(x)$ are polynomials, the degree of $P(x)$ being less than that of $Q(x)$. Let $a_i, i = 1, \dots, m$ be the roots of the polynomial $Q(x)$, and n_i be their corresponding multiplicities. Then

$$\frac{P(x)}{Q(x)} = \sum_{k=1}^{n_1} \frac{A_k^{(1)}}{(x-a_1)^k} + \sum_{k=1}^{n_2} \frac{A_k^{(2)}}{(x-a_2)^k} + \dots + \sum_{k=1}^{n_m} \frac{A_k^{(m)}}{(x-a_m)^k}$$

where $A_k^{(i)} = \frac{1}{(n_i - k)!} \frac{d^{n_i - k}}{dx^{n_i - k}} \left[\frac{P(x)}{Q(x)} (x-a_i)^{n_i} \right] \Bigg|_{x=a_i}$

If $n_i = 1$ for all, then

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{m} \frac{A(i)}{x - a_i}$$

where $A^{(i)} = P(a_i)/Q'(a_i)$.

When pairs of terms corresponding to complex conjugate roots are combined, one obtains fractions of the form $\frac{Ax+B}{x^2+bx+c}$, where the trinomial x^2+bx+c has real coefficients. Therefore the calculation of the integral $\int \frac{P(x)}{Q(x)} dx$ reduces to the integration of rational functions. $\int \frac{dx}{(x-a)^n}$, $\int \frac{Ax+B}{(x^2+bx+c)^n} dx$

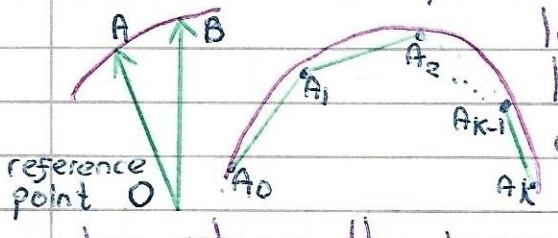
Curve Arc-Length

Curve Arc-Length The arc-length from t_0 to T of a curve $t \mapsto r(t)$, for which $r'(t)$ is continuous, is

$$\int_{t_0}^{T} |r'(t)| \, dt = \int_{t_0}^{T} \sqrt{x'(t)^2 + y'(t)^2} \, dt$$

We assume that $t \mapsto r(t)$ is injective on the domain $[t_0, T]$ with the possible exception of multiple-points.

We consider polygonal arcs from $r(t_0)$ to $r(T)$ all of whose vertices $A_1 = r(t_1), A_2 = r(t_2), \dots A_K = r(t_K)$ lie on the curve, where $t_0 < t_1 < \dots < t_K = T$. The arc length of a smooth curve is the limit, in the case where the limit exists, of the length of these polygonal arcs as the length of the



largest of the increments $t_i + 1 - t_i$, tends to zero. Given that a curve is a smooth curve, or that $r'(t)$ is continuous, it can be shown that the limit of the lengths of the arcs exists, and is equal to the quantity given by the above equation.

ODE's Review

Vectors

Integrating Factor

First Order Linear Differential Equation

The general form of a first order linear differential equation can be written as $y' + a(x)y = b(x)$ where $a(x)$ and $b(x)$ are known functions.

Multiplying both sides of the equation by $\exp[\int a(x)dx]$, which is often called the integrating factor, we have

$$y' e^{\int a(x)dx} + a(x) y e^{\int a(x)dx} = b(x) e^{\int a(x)dx}$$

which can be written as $\left[y e^{\int a(x)dx} \right]' = b(x) e^{\int a(x)dx}$

By simple integration we have $y e^{\int a(x)dx} = \int b(x) e^{\int a(x)dx} dx + C$

So it's solution becomes

$$y(x) = e^{-\int a(x)dx} \int b(x) e^{\int a(x)dx} dx + C e^{-\int a(x)dx}$$

where C is the integration constant.

Linear System of order 'n'

Linear System of order 'n'

$$a_n y_c^{(n)} + a_{n-1} y_c^{(n-1)} + \dots + a_1 y_c' + a_0 = 0$$

For a simple homogeneous system

$\frac{du}{dt} = \alpha u + \beta w$, $\frac{dw}{dt} = \gamma w$, we can write it as

$$\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

Here we assume the parameters α, β and γ are real constants.

As the matrix $A = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$ is a triangular matrix, its eigenvalues are the diagonal elements.

So the eigenvalues of A are $\lambda_1 = \alpha$, $\lambda_2 = \gamma$.

Following the procedure for finding the eigenvectors we have the eigenvector $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $\lambda_1 = \alpha$.

Similary the eigenvector for $\lambda_2 = \gamma$ is

$$V_2 = \frac{1}{\sqrt{1 + \frac{(\alpha - \gamma)^2}{\beta^2}}} \begin{pmatrix} 1 \\ -(\alpha - \gamma)/\beta \end{pmatrix}$$

The general solution of this linear system is $\begin{pmatrix} u \\ w \end{pmatrix} = k_1 V_1 e^{\alpha t} + k_2 V_2 e^{\gamma t}$

where k_1 and k_2 are constants.

Matrices

- $n \times n$ matrix $A = (a_{i,j})$ i = row, j = column

$$mat = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad mat = \begin{pmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{pmatrix}$$

- product $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ | multiplying gives $\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ -5 & 8 \end{pmatrix}$

- Symmetric where transpose = original, mat = mat'
  eg $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric

- Determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

note $|AB| = |A||B|$ and a nonsingular matrix is $|A| \neq 0$.

- Invertability Matrix is invertable if there is a matrix 'B' so that AB = the identity matrix. Also $BA = I$ (Identity matrix)
The inverse is denoted $^{-1}$ and satisfys $AA^{-1} = A^{-1}A = I$
The inverse exists only if $|A| \neq 0$. (Determinant).

- Also an Orthogonal Matrix is one where $A\,' = I$ or $A'\; = I$. A rotation matrix is an orthogonal that |orthogonal matrix| = 1 (Determinant)

- Note A' and B' are the transpose matrices and generally (outside of matlab) written A^T and B^T,

- Diagonal Matrix Where all entries are zero except those in the i^{th} row and i^{th} columns. denoted 'λ' where the 'i^{th}' entries are quoted as : $\lambda - \lambda_1$, λ_2 etc the matrix normally has letter Λ

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Line Integrals

Suppose $F = -3x^2 i + 5xy j$ and let C be the curve $y = 2x^2$ in the xy plane. Evaluate the line integral in terms of limits of sums:

$$\int_C F \cdot dr = \int_C (-3x^2 i + 5xy j) \cdot (dx\, i + dx\, j) = \int_C (-3x^2 dx + 5xy\, dy)$$

1st method $\int_C F \cdot dr = \int_{t=0}^{t} [-3t^2\, dt + 5t(2t^2)\, d(2t^2)] =$

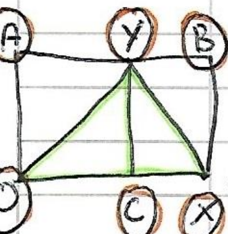
$$\int_{t=0}^{1} (-3t^2 + 40t^4)\, dt = [-t^3 + 8t^5]_0^1 = 7$$

Super Equations 1

$$\cos\theta = \frac{x \cdot y}{|x|\,|y|} \qquad x \cdot y = |x|\,|y| \cos\theta$$

$$\boxed{\text{Area triangle } \Delta XOY = \frac{\sqrt{|x|^2|y|^2 - (x \cdot y)^2}}{2}}$$

area $= \frac{1}{2}$ OX × height $= \frac{1}{2}|x|\,|y| \sin\theta$



$$= \frac{1}{2}|x|\,|y| \sqrt{1 - \left(\frac{x \cdot y}{|x|\,|y|}\right)^2} = \frac{\sqrt{|x|^2|y|^2 - (x \cdot y)^2}}{2}$$

area general triangle $\Delta XYZ = \dfrac{\sqrt{|X-Z|^2\,|Y-Z|^2 - ((X-Z)\cdot(Y-Z))^2}}{2}$

Slope line $\dfrac{y_1 - y_2}{x_1 - x_2}$ (origin)

angle two lines through origin $\tan\alpha = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

Hyperplane normal equation $ax + by + cz = d$
Solve gaussian style eq for a plane with points $(1,2,3)$ and $(2,0,5)$ would be $a + 2b + 3c = d$, $2a + 5c = d$

α Nearest point to hyperplane $A.X = b$ is $\dfrac{b}{|A|^2} A$
distance from origin $\dfrac{|b|}{|A|}$

Nearest point on hyperplane to point P is $\dfrac{b - A \cdot P}{|A|^2} A + P$
distance from point P $\dfrac{|b - A \cdot P|}{|A|}$

θ Rotation/Translation of ~~an~~ points on one set of axes to another through angle α about the origin.
(s, t) = new coordinates, α = angle about origin to new set, (x, y) = current coordinates.

$$s = x\cos\alpha + y\sin\alpha$$
$$t = x\sin\alpha - y\cos\alpha$$

On substituting x and y we have

$$ax^2 + 2hxy + vy^2 = a's^2 + 2h'st + b't^2.$$

Next Section - Vectors